PRICE VS. RESERVE REGULATION CONDITIONED BY SOLVENCY REQUIREMENTS IN THE COLLECTIVE RISK MODEL

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ABSTRACT

The policy prices adjusted vs. reserves conditioned by solvency in a short term time horizon are considered. The motivation is to step towards considering insurer as subject of price competitive insurance market.
The intrinsic relationship between policy prices and reserves and its influence on solvency of individual insurance business are formalized in the framework of the collective risk model. Different approaches to tuning prices vs. reserves conditioned by solvency requirements expressed in terms of the probability of ruin within finite time and of the ultimate ruin probability, based on (a) exact numerical technique, (b) new approximations, and (c) simulation, are discussed.

KEYWORDS

Solvency of insurer. Capital requirements. Price competition. Finite time and ultimate ruin probabilities.

1. INTRODUCTION

Insurance is a method of coping with risks, while the object of the theory of risk is to give a mathematical analysis of random fluctuations in an insurance business and to discuss the various means of protection against their inconvenient effects.
The insurer is subject of competitive insurance market where the policy prices are among the primary influences. The prices, the reserves, and solvency of each individual business are inseparable. Both practical and theoretical aspects of the solvency of insurer and the respective price and capital requirements are clearly in evidence (see e.g., Section 14.6 of [9]). Developing the system of solvency testing, the theory of risk is applied, bringing to the forefront either analytical or simulation approach.
The traditional models of the risk theory have certain recognized shortcomings which arise from the structure deficiencies rather than merely from the technical restrictions, though the latter attracted more attention in literature. One of these shortcomings is making no sufficient allowance for the interdependence between the premium rate and the capital size which makes some important business aspects overlooked, even as the attention is focused on testing the financial position of insurer at the annual interval and inflation, return on investments, market cycles, and certain other premises are allowed to be neglected.
In this paper we discuss a generalization of the collective risk model (see [12], [13]), binding together the initial capital and the risk premium rate so that the relative safety loading becomes decreasing, as the initial capital grows. Unlike making the risk premium rate variable and dependent on the current value of the risk reserve (see e.g., [3], [4], [5]), our approach reflects the fact that supervision is usually implemented by testing the financial position of each insurer at regular intervals, normally annually. It might rouse the company to scheduled actions such as annual change of prices and reserves. The disadvantages of such a "scheduled" price vs. solvency optimization is substantially compensated by solvency requirements expressed in terms of the finite time ruin probabilities which in a sense preserves the risk reserve from being too small frequently in between the accounting moments.
In Section 2 we revise an example by Seal, applying exact numerical techniques. Our consequential aim is to impeach his contention formulated in [15], [16] which regards this example as "an emphatic illustration of the poverty of asymptotic numerical approximations for the practical man" and to present the adequate asymptotic results.
In Section 3 we develop examination by Seal and demonstrate crude and delicate tuning of price vs. reserve conditioned by solvency.
In Section 4 we revise an exact numerical technique applied to probabilities of ruin within finite time.

In Section 5 we suggest an adjustment of the collective risk model where the safety loading τ_u depends on the initial capital u and propose new approximations for the probabilities of ruin, as u increases. Unlike the standard Cramér-Lundberg approximations, they are suitable for the framework considered in Sections 2 and 3. It resolves the shortcoming perceived by Seal.
Though approximations are more flexible than the exact numerical technique, we are fully sharing the opinion that any formula approach inevitably has rigid limitations. A reputed contender of the formula approach is simulation. In Section 6 we briefly consider the stochastic bundle approach described in Section 14.6 of [9] and point out several complications emerging in the modified risk model.

2. AN OBSERVATION BY SEAL

The theory of risk focuses its attention on the reference insurer through the outflow process, looking first at claim numbers, then at the distribution of claim sizes and finally putting these two together into an aggregate claim amounts process. The income process, which is the initial capital plus premium income, is introduced in a rather simple way, growing linearly in time with a constant intensity c. The resulting surplus process of the insurance business is generated as initial capital plus premium income minus outflow.
This bird's eye view has been formalized in the notion of the collective risk model which remains up to now one of the main actuarial premises concerned with final business results. Ignoring individual policies, this model views an insurance business as a whole: claims occur from time to time and are settled by the company, while on the other hand the company receives a continuous flow of risk premiums from the policyholders.
Mathematically, the surplus process at any operational time t is described by the risk reserve process $R(t) = u - \sum_{i=1}^{N(t)} Y_i + ct$ starting at time $t = 0$, where $N(t)$ is the number of claims occurred up to time t, $u > 0$ is the initial risk reserve. The insurance company is supposed to pay premiums as they are received into a risk reserve and $c > 0$ is the risk premium rate, $\{T_i\}_{i\geqslant 1}$ are (i.i.d.) interclaim times and $\{Y_i\}_{i\geqslant 1}$ are (i.i.d.) amounts of claims.
Supervision is usually implemented by testing the financial position of each insurer at regular intervals, normally annually. In practice, adverse fluctuations often occur in consecutive years, giving rise to considerable accumulation of losses. This may not be revealed by an analysis limited to one calendar year. But the short term time horizon, in most cases one year, is the fundamental building block for the long-term analysis.
Mathematically, the probability of ruin $\psi(t,u) = \mathbf{P}\{\inf_{0<s\leqslant t} R(s) < 0\}$ within the time interval $(0,t]$, which particular case is the probability of ultimate ruin $\psi(u) = \psi(+\infty, u)$, is an important scientific paradigm: within the collective model the solvency requirement are similar to the restrain to have the probability of ruin at a certain prescribed level all along the accounting period.
Two reputed influences are as follows. First, the insurer typically needs to charge *loaded* premiums sufficient for business to take its normal course over a long time. The amount $\tau = c\mathbf{E}T_1/\mathbf{E}Y_1 - 1$, called the relative safety loading, reflects this need. Indeed, since cT_i is the premium acquired and Y_i is the claims amount paid out on the i-th "step" which is the time between $(i-1)$-th and i-th claims, the condition $\tau > 0$ means that successful "steps" are persistent. The opposite, $\tau < 0$, means that successful "steps" are rare and, in total, the business is trading unfavorably and is liable to ruin. Second, the insurers are required by law to keep the necessary *reserves* to safeguard solvency and, in particular, to meet early claims.
Acting on a competitive insurance market, the insurer might be interested to establish a balance of price (by decreasing the loading τ) vs. reserve u, aiming legal or desired level of solvency expressed by the probability $\psi(t,u)$ within the accounting period $(0,t]$.
To our opinion, an early attempt to examine this balance was made by Seal. In [15] he considers the collective risk model where exponential claims are occurring as a Poisson process. He takes unit Poisson intensity, so that the unit of time is the expected interval between claim occurences, and unit exponential distribution parameter, so that the average individual claim size is the monetary unit. In [16] he sharpens his outlines by considering constant unit claims.
Basing on an exact formula which Seal attributed to Arfwedson [1], for no-loading $\tau = 0$ and for $\tau = 0.1$ which is a 10% risk loading in this model, he calculates numerically the probability of non-ruin within the interval $(0,t]$, $U(t,u) = \mathbf{P}\{\inf_{0<s\leqslant t} R(s) \geqslant 0\}$.
Seal's analysis (see [16], pp. 128 – 129; we adapted it to suit the model in [15]) of the calculations which we partly reproduced from [15], Tables 2 and 3, in our Tables 1 and 2, is as follows: "with no risk loading — which is known to lead to ultimate ruin whatever (finite) value u has — and an initial risk-reserve of as little as ten times the mean unit claim there is still an 70.5% chance

of not being ruined during an interval within which 50 claims are expected." "One sees how far 50 is from infinity so far as the probability of ruin is concerned!" exclaimed Seal. Table 2 indicates that with u as large as 110 there is a 99.8% chance of not being ruined in an interval during which 600 claims are expected.
Seal's conclusion is that it is an "emphatic illustrations of the poverty of asymptotic numerical approximations for the practical man". He writes: "The real value of risk theory is, we believe, to the *entrepreneur* just starting out in business with a casualty insurance company. It is the early claims that worry him not those that occur after he has built a successful business!"

Table 1. Probability of non-ruin $U(t,u)$ for Poisson-exponential model, $\tau = 0.1$

	u										
t	0	1	2	3	4	5	6	7	8	9	10
1	0.53660	0.76194	0.88029	0.94085	0.97121	0.98616	0.99342	0.99690	0.99855	0.99933	0.99969
5	0.28040	0.48811	0.64558	0.76049	0.84164	0.89734	0.93464	0.95906	0.97474	0.98463	0.99077
10	0.21457	0.38742	0.53087	0.64690	0.73857	0.80943	0.86312	0.90305	0.93224	0.95323	0.96810
20	0.16816	0.30939	0.43267	0.53879	0.62889	0.70438	0.76683	0.81785	0.85904	0.89191	0.91785
30	0.14798	0.27393	0.38578	0.48419	0.57000	0.64413	0.70760	0.76147	0.80678	0.84458	0.87584
40	0.13621	0.25289	0.35738	0.45033	0.53247	0.60458	0.66744	0.72188	0.76871	0.80872	0.84269
50	0.12836	0.23872	0.33804	0.42696	0.50618	0.57639	0.63827	0.69253	0.73985	0.78090	0.81631
t	0	11	22	33	44	55	66	77	88	99	110
50	0.12836	0.84671	0.98438	0.99904	0.99996	1.0	1.0	1.0	1.0	1.0	1.0
100	0.11001	0.77244	0.95621	0.99373	0.99933	0.99991	1.0	1.0	1.0	1.0	1.0
150	0.10282	0.73611	0.93517	0.98695	0.99786	0.99971	0.99997	1.0	1.0	1.0	1.0
200	0.09902	0.71512	0.92050	0.98080	0.99602	0.99929	0.99989	0.99998	0.99999	1.0	1.0
400	0.09343	0.68177	0.89287	0.96584	0.98979	0.99716	0.99927	0.99982	0.99994	0.99998	0.99999
600	0.09191	0.67215	0.88372	0.95977	0.98652	0.99565	0.99865	0.99960	0.99986	0.99996	0.99998
∞	0.09091	0.66556	0.87697	0.95474	0.98335	0.99387	0.99775	0.99917	0.99970	0.99989	0.99996

Table 2. Probability of non-ruin $U(t,u)$ for Poisson-exponential model and no risk loading

	u										
t	0	1	2	3	4	5	6	7	8	9	10
1	0.52378	0.75406	0.87580	0.93842	0.96993	0.98551	0.99309	0.99674	0.99848	0.99929	0.99967
5	0.24910	0.45252	0.61280	0.73344	0.82085	0.88216	0.92399	0.95183	0.96996	0.98154	0.98881
10	0.17729	0.33697	0.47678	0.59522	0.69263	0.77066	0.83168	0.87837	0.91338	0.93916	0.95782
20	0.12576	0.24501	0.35614	0.45764	0.54860	0.62869	0.69804	0.75713	0.80675	0.84783	0.88137
30	0.10279	0.20198	0.29653	0.38535	0.46767	0.54295	0.61092	0.67157	0.72505	0.77168	0.81191
40	0.08907	0.17577	0.25939	0.33912	0.41433	0.48454	0.54942	0.60878	0.66260	0.71093	0.75395
50	0.07969	0.15768	0.23343	0.30638	0.37584	0.44158	0.50321	0.56053	0.61341	0.66181	0.70578
t	0	11	22	33	44	55	66	77	88	99	110
50	0.07969	0.74543	0.96330	0.99701	0.99985	1.0	1.0	1.0	1.0	1.0	1.0
100	0.05638	0.59118	0.87760	0.97439	0.99617	0.99958	0.99997	1.0	1.0	1.0	1.0
150	0.04605	0.50370	0.80017	0.93780	0.98492	0.99712	0.99957	0.99995	1.0	1.0	1.0
200	0.03988	0.44602	0.73716	0.89789	0.96746	0.99145	0.99813	0.99966	0.99995	0.99999	1.0
400	0.02821	0.32649	0.57755	0.76124	0.87868	0.94462	0.97728	0.99161	0.99721	0.99916	0.99977
600	0.02303	0.26976	0.48941	0.66709	0.79802	0.88612	0.94037	0.97100	0.98690	0.99450	0.99786
∞	always zero for finite u										

3. PRICE VS. RESERVE BALANCE IN THE RISK MODEL: AN EXAMPLE

To our opinion, a sensible extension of the considerations by Seal are crude and delicate tuning of price (or rather its safety loading component which is flexible) vs. reserve conditioned by solvency expressed through the ruin probabilities. The former consists in the gradual reduction of the safety loading roughly sketched as passing from Table 1 to Table 2, looking back at $\psi(u)$ and at u, the later consists in finer time considerations, since the magnitude of u and τ is roughly chosen, and is based on more insight at $\psi(t,u)$.

Our first concern is an example of what we called crude tuning. As Seal in [15], consider the Poisson-exponential risk model with unit Poisson intensity and unit parameter of the exponential claim size distribution. Assume rather arbitrarily the starting value of the initial risk reserve to be $u_0 = 10$ and the respective value of the ultimate ruin probability to be $\psi(u_0) = 0.1$ (in practice they could be those values which have been applied on the preceding accounting period). Evidently, using the explicit formula (1) below, the safety loading calculated numerically must be $\tau = 0.26113$.

The actuary is faced the problem to analyze the following balance: to decrease the safety loading (i.e., to reduce prices) respecting solvency requirements e.g., to keep the ultimate ruin probability between 0.1 and 0.05 (too small values might be considered superfluous and unrealistic), by means of a sensible increase of the initial risk reserve u, say up to the values in between $u = 20$ and $u = 60$.

Once the exact result like (1) below is available, one easily constructs a lower bound for τ as function of u: for each value of u one should solve the equation $(1+\tau)^{-1}\exp(-u\tau(1+\tau)^{-1}) = 0.05$ w.r.t. τ. But the rate of decrease of τ, as u increases, appears inscrutable. Moreover, the link between the lower bound and the initial values which were $u_0 = 10$ and $\psi(u_0) = 0.1$ in our example, are rather implicit. This approach appears highly sensible to any deviation from the original model assumptions.

The standpoint might be revised: one may assign certain parametric families e.g., $\tau_u = au^{-k}$, or $\tau_u = (\ln u)^{-k}$, $k > 0$, $a > 0$, which mirror more or less aggressive price policy, aiming evaluation of the parameters of these families according to solvency and capital requirements. To our opinion, this approach sheds more light at the process of decision making. Fortunately, it beautifully complies with the refined asymptotical approach, as u increases, which will be developed for the general claim size and interclaim times distributions in Section 5.

O [illegible] $= 0.68158\,u^{-5/12}$,

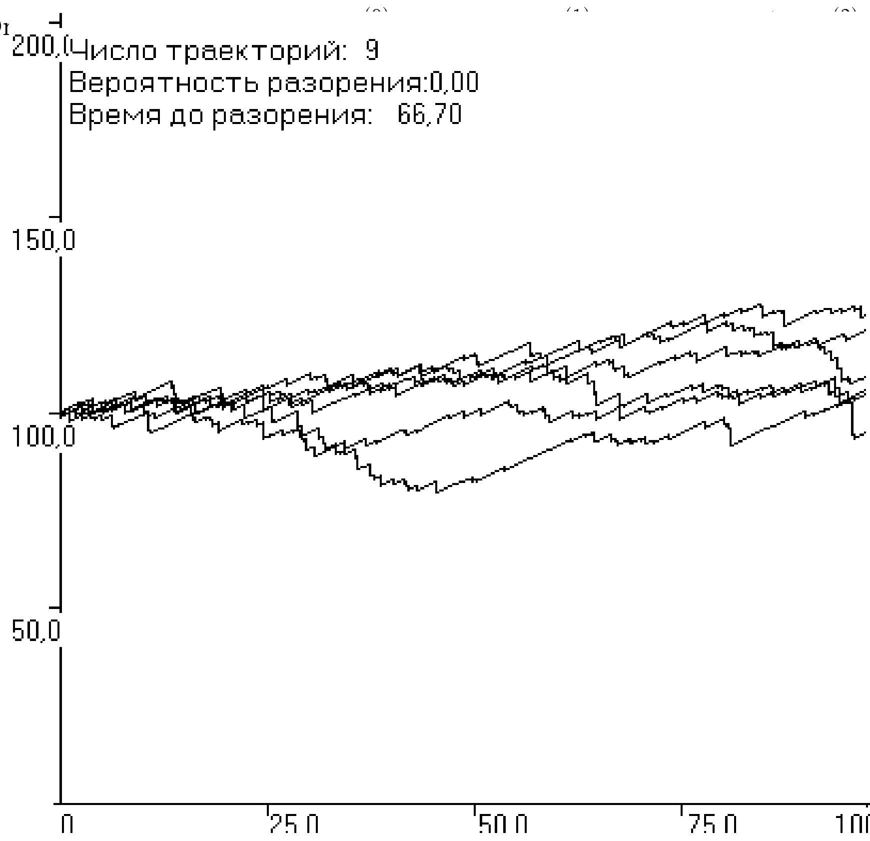

Fig. 1. Constant safety loading $\tau_u^{(0)} = 0.26113$ (thick line), variable safety loadings $\tau_u^{(1)} = 0.31636\,u^{-1/12}$ (dashed line with long segments), $\tau_u^{(2)} = 0.68158\,u^{-5/12}$, $\tau_u^{(3)} = 1.46842\,u^{-9/12}$, $\tau_u^{(4)} = 2.15534\,u^{-11/12}$ (prick line) and the ultimate ruin probabilities $\psi_k(u) = (1+\tau_u^{(k)})^{-1}\exp(-u\tau_u^{(k)}(1+\tau_u^{(k)})^{-1})$, conditioned by $\psi_k(10) = 0.1$, $k = 0, 1, 2, 3, 4$.

$\tau_u^{(3)} = 1.46842\,u^{-9/12}$, $\tau_u^{(4)} = 2.15534\,u^{-11/12}$ and the respective ultimate ruin probabilities $\psi_k(u) = (1+\tau_u^{(k)})^{-1}\exp(-u\tau_u^{(k)}(1+\tau_u^{(k)})^{-1})$, conditioned by $\psi_k(10) = 0.1$, $k = 0,1,2,3,4$. One sees that only the rate $\tau_u^{(4)}$ is nearly satisfying our original requirements. However, if the solvency requirements will be lowered at 0.025 the attention should be paid to $\tau_u^{(3)}$.
Delicate tuning refines these considerations allowing for finitude of the time interval. Indeed, we based our previous considerations on the ultimate ruin probability $\psi(u)$ which is merely an upper bound for $\psi(t,u)$, whenever finite time t is concerned. This idea was basic in Section 2 when safety loading was assumed zero but non-ruin remained rather probable for certain finite t.
On Fig. 2 below we present the probabilities $\psi(t,u)$ of ruin within finite time t for the rates $\tau_u^{(2)} = 0.68158\,u^{-5/12}$ and $\tau_u^{(3)} = 1.46842\,u^{-9/12}$, and for $u = 20$ and 30, calculated numerically by applying the exact formula (2) below. Evidently, $\psi_2(20) = 0.0317$, $\psi_2(30) = 0.0122$, $\psi_3(20) = 0.0589$, $\psi_3(30) = 0.0589$ will be considerably undershot if e.g., $t = 200$.

4. EXACT NUMERICAL METHODS

The implementations of Seal and the examples of Sections 2 and 3 are crucially based on the exact numerical methods. Seal was based on a numerical integration worked out for the Poisson claims arrival. It is described in details in his books [14], [17]. The calculations of Section 3 apply the famous Cramér exact formula (1), and the formula (2) which can be found for $\mu = c = 1$ in [2], [7], and in [11].

Assertion. *In the Poisson-exponential risk model*

$$\psi(u) = \frac{\lambda}{c\mu}\exp(-u(c\mu-\lambda)/c) \tag{1}$$

and

$$\psi(t,u) = \psi(u) - \frac{1}{\pi}\int_0^\pi f(x\,;\,t,u)dx \tag{2}$$

for any $u > 0$, where

$$\begin{aligned} f(x\,;\,t,u) = (\lambda/c\mu)&(1+\lambda/c\mu - 2\sqrt{\lambda/c\mu}\cos x)^{-1} \\ &\times \exp\left(u\left(\sqrt{\lambda\mu/c}\cos x - 1\right) + tc\mu\left(2\sqrt{\lambda/c\mu}\cos x - \lambda/c\mu - 1\right)\right) \\ &\qquad\qquad \times \left[\cos\left(u\sqrt{\lambda\mu/c}\sin x\right) - \cos\left(u\sqrt{\lambda\mu/c}\sin x + 2x\right)\right]. \end{aligned}$$

It is worth mentioning that (2) was derived in [11] as a corollary of the following result which supplies an exact numerical technique for a non-Poisson claims arrival.

Theorem 1. *Let the sizes of claims $\{Y_i\}_{i\geqslant 1}$ and the interclaims times $\{T_i\}_{i\geqslant 1}$ be i.i.d. and mutually independent. Let Y_1 be exponential with a positive parameter μ and the Laplace transform of T_1 be $\gamma(\alpha) = \mathbf{E}e^{-\alpha T_1}$, $\alpha > 0$. Then for any $u > 0$*

$$\alpha\int_0^\infty e^{-\alpha t}\psi(t,u)\,dt = y(\alpha)\exp\{-u\mu(1-y(\alpha))\}, \quad \alpha > 0,$$

where $y(\alpha)$ is a solution of the equation

$$y(\alpha) = \gamma(\alpha + c\mu(1-y(\alpha))), \quad \alpha > 0.$$

5. APPROXIMATIONS

Any formula approach inevitably has its limitations. The limits of the exact methods are particularly tights. Therefore the risk problem was attacked by asymptotic methods. The most famous are the Cramér-Lundberg approximations like

$$\lim_{u\to\infty} e^{\varkappa u}\psi(u) = C, \tag{3}$$

where C is the Cramér-Lundberg constant, and

$$\lim_{u\to\infty}\sup_{t\geqslant 0}|\psi(t,u)e^{\varkappa u}-C\,\Phi_{(mu,D^2u)}(t)|=0\,, \tag{4}$$

where $\Phi_{(mu,D^2u)}(t)$ is the Normal probability distribution function (see e.g., [6]).
Going back to the Section 2, the discontent of Seal who declared poverty of the asymptotic numerical approximations clears up when we conceive the major restriction of (3) and (4) which in fact is that we must have c constant, as u is growing. It means e.g., that we are allowed to apply (3) and (4) for approximations of the thick line on Fig. 1, but not allowed to apply these results for approximation of dashed ones. To our opinion, Seal pointed implicitly this gap between asymptotical and exact numerical methods by an extreme case of no-loading, or extremely small c, and rather large u. However, we blame certain deficiency of a particular risk model, rather than asymptotic approximations approach per se. As a conclusion, we have to refine the collective risk model and to extend the approximations (3) and (4) to make them valid within this new, refined, model.
Introduce $\tau_u>0$ depending on u and such that $\tau_u\to 0$ monotonously, as u grows to infinity, starting from a certain positive value u_0. The particular choice of τ_u depends on external factors and the motivation deserves a separate discussion outside the scope of this paper.
For i.i.d. random vectors (T_i,Y_i), $i=1,2,\dots$, define a series of the risk reserve processes $R_u(t)=u-\sum_{i=1}^{N(t)}Y_i+c_ut$, where the premium rate is $c_u=(1+\tau_u)\mathbf{E}Y_1/\mathbf{E}T_1$. For $i=1,2,\dots$ introduce i.i.d. random variables $X_{u,i}=Y_i-c_uT_i$ and put $S_{n,u}=\sum_{i=1}^n X_{u,i}$, $U_n=\sum_{i=1}^n T_i$. For the p.d.f. $B_u(x,y)=\mathbf{P}\{X_{u,1}\leqslant x,T_1\leqslant y\}$ and for a positive solution $\varkappa_u$ of the Lundberg equation

$$\mathbf{E}\exp(\varkappa_u X_{u,1})=1$$

introduce an associate p.d.f. by $\overline{B}_u(dx,dy)=e^{\varkappa_u x}B_u(dx,dy)$. Introduce the associated sequence $\{(\overline{X}_{u,i},\overline{T}_{u,i})\}_{i\geqslant 1}$ of i.i.d. random vectors having the p.d.f. $\overline{B}_u(x,y)$, and $\overline{S}_{n,u}=\sum_{i=1}^n\overline{X}_{u,i}$, $\overline{U}_{n,u}=\sum_{i=1}^n\overline{T}_{u,i}$. Put

$$\overline{\beta}_u(t_1,t_2)=\iint e^{i(t_1x+t_2y)}\overline{B}_u(dx,dy),$$

$$\rho_u(t_1,t_2)=\sum_{n=1}^{\infty}\frac{1}{n}\iint_{x\leqslant 0}e^{i(t_1x+t_2y)}\overline{B}_u^{*n}(dx,dy),$$

$$\nu_u^{i,j}=\mathbf{E}X_{u,1}^iT_1^j,\ \overline{\nu}_u^{i,j}=\mathbf{E}\overline{X}_{u,1}^i\overline{T}_1^j,\ i,j=0,1\dots\,.$$

As usual, asterisk denotes convolution. Introduce

$$m_u=\overline{\nu}_u^{0,1}/\overline{\nu}_u^{1,0},\quad D_u^2=((\overline{\nu}_u^{0,1})^2\overline{\nu}_u^{2,0}-2\overline{\nu}_u^{1,0}\overline{\nu}_u^{0,1}\overline{\nu}_u^{1,1}+(\overline{\nu}_u^{1,0})^2\overline{\nu}_u^{0,2})/(\overline{\nu}_u^{1,0})^3,$$

$$C_u=\frac{1}{\varkappa_u\overline{\nu}_u^{1,0}}\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}\big[\mathbf{P}(S_{n,u}>0)+\mathbf{P}(\overline{S}_{n,u}\leqslant 0)\big]\Big).$$

In the aggregate with the approximations $\varkappa_u=\mathfrak{a}_1\tau_u+\mathfrak{a}_2\tau_u^2+\dots$, $m_u=\mathfrak{m}_{-1}\tau_u^{-1}+\mathfrak{m}_0+\mathfrak{m}_1\tau_u+\dots$, $D_u^2=\mathfrak{d}_{-3}\tau_u^{-3}+\mathfrak{d}_{-2}\tau_u^{-2}+\mathfrak{d}_{-1}\tau_u^{-1}+\dots$, and $C_u=1+\mathfrak{c}_1\tau_u+\dots$, as $u\to\infty$ (see Theorems 3.1, 3.2, 3.3, and 4.1 in [12]), this "scheme of series" counterpart of the approximations (3) and (4) constitutes the main result of this Section.

THEOREM 2. *In the risk model with the p.d.f. B_{YT} having a bounded density w.r.t. Lebesgue measure on R^2, assume that $c_u=(1+\tau_u)\mathbf{E}Y_1/\mathbf{E}T_1$ with $\tau_u\geqslant u^{-5/12}$ and for a sufficiently large $u_0>0$*

1. $\sup_{u>u_0}\iint_{-\infty}^{\infty}|\overline{\beta}_u(t_1,t_2)|^p dt_1dt_2<\infty$, $\sup_{u>u_0}\iint_{-\infty}^{\infty}|\rho_u(t_1,t_2)|^p dt_1dt_2<\infty$ *for some* $p\geqslant 1$,
2. $\sup_{u>u_0}\mathbf{E}e^{a|X_{u,1}|}<\infty$, $\sup_{u>u_0}\mathbf{E}e^{a|\overline{X}_{u,1}|}<\infty$ *for some* $a>0$,
3. $D^2=\lim_{u\to\infty}D_u^2>0$.

Then

$$\sup_{t\geqslant 0}\left|\psi(t,u)-C_u e^{-\varkappa_u u}\Phi_{(m_u u,\,D_u^2 u)}(t)\right| = \underline{O}\big((\tau_u u)^{-1/2}e^{-\varkappa_u u}\big)\,, \tag{5}$$

as $u\to\infty$.

The following result is a corollary of Theorem 2.

THEOREM 3. *In the Poisson-exponential risk model assume that* $c_u=(1+\tau_u)\mathbf{E}Y_1/\mathbf{E}T_1$ *with* $\tau_u\geqslant u^{-5/12}$. *Then*

$$\sup_{t\geqslant 0}\left|\psi(t,u)-C_u e^{-\varkappa_u u}\Phi_{(m_u u,\,D_u^2 u)}(t)\right| = \underline{O}\big((\tau_u u)^{-1/2}e^{-\varkappa_u u}\big)\,, \tag{6}$$

as $u\to\infty$, *where* $\varkappa_u=\mu\tau_u/(1+\tau_u)$, $m_u=\mu/(\lambda\tau_u(1+\tau_u))$, $D_u^2=2\mu/(\lambda^2\tau_u^3)$, *and* $C_u=1/(1+\tau_u)$.

Unlike (3) and (4), (5) and (6) are suitable for the approximation of the dashed lines on Fig. 1. We bound ourselves by a numerical example where the finite ruin probabilities $\psi(t,u)$ are concerned, comparing the exact values calculated numerically by means of (2) and the approximations calculated by means of (6) for $\tau_u^{(2)}=0.68158\,u^{-5/12}$ and $\tau_u^{(3)}=1.46842\,u^{-9/12}$, and $u=20$ and 30.

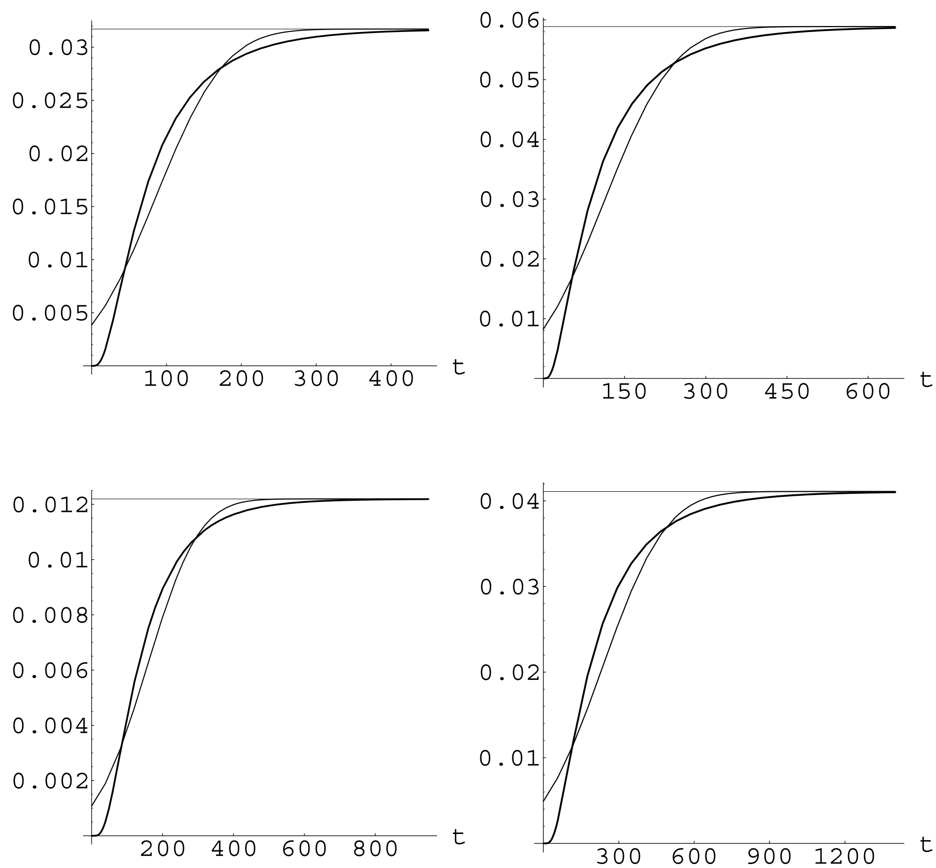

Fig. 2. Exact values of $\psi(t,u)$ (lines starting from origin) and the approximations $\psi_{[appr]}(t,u)=C_u e^{-\varkappa_u u}$ $\Phi_{(m_u u,\,D_u^2 u)}(t)$ for $\tau_u^{(2)}=0.68158\,u^{-5/12}$, $u=20$ (upper left) and $u=30$ (lower left), and for $\tau_u^{(3)}=1.46842\,u^{-9/12}$, $u=20$ (upper right) and $u=30$ (lower right), $\lambda=\mu=1$.

It is worth noting that if a better accuracy is desired, as u grows, (5) and (6) should be refined e.g., by the asymptotic expansions like in [10]. The same should be done to restore the approximations force if only moderate or even small values of u are available.
We finish this Section by a remark that Theorem 2 is suitable for various generalizations of the Poisson-exponential model where there are no exact formulas like (1) and (2), and even no exact numerical technique as in Theorem 1, or in the books [14] and [17].
Let the (i.i.d.) amounts of claims $\{Y_i\}_{i\geqslant 1}$ be independent on the (i.i.d.) inter-occurrence times $\{T_i\}_{i\geqslant 1}$. Let Y_1 be Gamma with the shape parameter β and the scale parameter μ, T_1 be Gamma with the shape parameter α and the scale parameter λ, all these parameters being positive. Then (see Example 3.1 in [12]) $\mathfrak{a}_1 = 2\alpha\mu/(\beta(\alpha+\beta))$, $\mathfrak{a}_2 = -4\alpha\mu(2\alpha+\beta)/(3\beta^2(\alpha+\beta)^2)$, $\mathfrak{a}_3 = 2\alpha\mu(14\alpha^2+17\alpha\beta+5\beta^2)/(9\beta^3(\alpha+\beta)^3)$, $\mathfrak{m}_{-1} = \alpha\mu/\lambda$, $\mathfrak{m}_0 = -2\mu\alpha(2\alpha+\beta)/(3\lambda\beta(\alpha+\beta))$, $\mathfrak{m}_1 = 2\mu\alpha(2\alpha+\beta)/(3\lambda\beta^2(\alpha+\beta))$, $\mathfrak{v}_{-3} = \alpha\beta(\alpha+\beta)\mu/\lambda^2$, $\mathfrak{v}_{-2} = 0$, which yields approximations to $\varkappa_u$, m_u, D_u^2, C_u. The conditions of Theorem 2 are satisfied and the approximation (5) is therefore valid.

6. SIMULATION

The idea of simulation approach in the standard risk model within finite time horizon t is based on simulation of the risk reserve as difference between incoming premiums and outgoing claims (see e.g., [9], Section 1.3, Fig. 1.3.2) and on consequent derivation of a stochastic bundle (see e.g., [9], Section 13.2, Fig. 13.1.5) comprising N independent risk reserve realizations. This direct approach can be refined e.g., by applying the importance sampling idea (see e.g., [3]), but the essence remains as above.
Daykin et al. wrote: "Once a ruin barrier has been defined, the simulated paths of the course of business which pass below the barrier are counted as ruin. Then the ratio of the number of ruins n_{ruin} to the total number N of realizations in the simulation gives an estimate of the probability of ruin. A visual inspection of the bundle of simulated paths can provide a good idea of the risk structure. Imagination can complete the bundle of the paths of the density configuration, which is illustrated in Fig. 13.1.4." ([9], p. 361).
In our case the ruin barrier is fixed and zero. Instead, which is equivalent when the premium rate c is constant and independent on u, one should choose u such that the ruin probability $\psi(t,u)$ remains within certain limits for a given c. The equivalence is due to the fact that when c is constant and independent on u the shape of the simulated bundle is invariant under shifting the initial capital up and down.
When the premium rate c depends on the initial capital u, as in Sections 3 and 5, shifting u means shifting c, which produces deformation of the entire bundle of simulated paths. The evaluation of the solvency margin by the "up and down shifting" of a single bundle of simulated paths gets no more possible and each new trial would require evaluation of the bundle of simulated path anew. It might increase dramatically the simulation complexity.

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