ZONE-ADAPTIVE CONTROL STRATEGY
FOR A MULTIPERIODIC MODEL OF RISK

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AGENDA

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1. Introduction: deficiency of traditional Risk Theory

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• Quotation from **H. Bohman'** (p. 2) farewell interview as retiring Chief Editor of the Scandinavian Actuarial Journal:

> "I was for a long time deeply involved in this theory, working on the probability of ruin, but I am hesitant over it now . . . From a practical point of view, the theory of collective risk, as initiated by Filip Lundberg, has missed the point, *because the underlying model is unrealistic, too simplified.* For one thing, a stationary business should give stationary reserves, as predicted by the control theory."

• Quotation from **C. Philipson** (p. 68):

> "From the development of the classical form [of the risk theory. — V.M.] two lines of development have branched out, one refers to the *generalization of the fundamental assumptions* . . . The other refers to the *extensions of the decision theory* . . . These lines of development are, however, all based on the fundamental conception of the collective risk theory, which was created by Filip Lundberg . . ."

- Quotation from **K. Borch** (p. 451):

  "We have now reached the point where the actuarial theory of risk again joins the mainstream of theoretical statistics and applied mathematics. Our general formulation of the actuary's problem leads directly to the general theory of *optimal control processes* or *adaptive control processes* . . .

  The theory of control processes seems to be "tailor–made" for the problems which actuaries have struggled to formulate for more than a century."

- Quotation from **C. D. Daykin, T. Pentikäinen, M. Pesonen** (Ch. 1, Sec. 5.5, p. 154):

  "It is worth mentioning that *the classical analytical methods and simulation should not be regarded as being in competition*. A general rule is that an *analytical technique should always be used wherever it is tractable*. On the other hand, the temptation should be resisted to manipulate the premises of the model in order to make the analytical calculations possible, if that can only be done at the cost of the applicability of the model to real–world conditions. If that is done, as is often the case in theoretically–orientated risk theory, a warning of the restricted applicability — or non–applicability — should be clearly given. The wide realm of application of simulation methods begins at the frontier where other methods become intractable."

2. Managing solvency: simulation analysis of insurance risk process, scenario-based DFA, EC Directives

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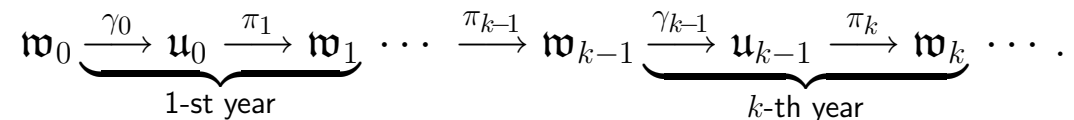
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3. Modelling of multiperiodic controlled insurance process

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Multiperiodic controlled insurance process

General multiperiodic insurance process with annual accounting and annual control interventions

$$\mathfrak{w}_0 \underbrace{\xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \mathfrak{w}_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \cdots .$$

According to this diagram (for $k = 1, 2, \ldots$), at the end of $(k-1)$-th year the **state variable** $\mathfrak{w}_{k-1}$ is observed. It describes the insurer's position at that moment. Then, at the beginning of k-th year the control rule γ_{k-1} is applied to choose the **control variable** $\mathfrak{u}_{k-1}$. Then k-th year probability mechanism of insurance unfolds; the transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k-th year.

Singleperiodic (annual generic) risk models

- *Diffusion*
- *Poisson–Exponential (classical)*

Scenarios of Nature

Random processes formalism

Set $\mathcal{Y}_0 = \{\mathfrak{w}_0\}$ and for $k = 2, 3, \ldots$ put

$$\mathcal{Y}_{k-1} = \{\mathfrak{w}_0, \ldots, \mathfrak{w}_{k-1}; \mathfrak{u}_0, \ldots, \mathfrak{u}_{k-2}\}$$

for the "history" up to the $(k-1)$-th year inclusively. Introduce

$$\pi_k(\mathcal{Y}_{k-1}, \mathfrak{u}_{k-1}; d\mathfrak{w}_k) = \pi_k(\mathfrak{w}_0, \ldots, \mathfrak{w}_{k-1}, \mathfrak{u}_0, \ldots, \mathfrak{u}_{k-1}; d\mathfrak{w}_k),$$

$$\gamma_{k-1}(\mathcal{Y}_{k-1}; d\mathfrak{u}_{k-1}) = \gamma_{k-1}(\mathfrak{w}_0, \ldots, \mathfrak{w}_{k-1}, \mathfrak{u}_0, \ldots, \mathfrak{u}_{k-2}; d\mathfrak{u}_{k-1})$$

called transition function of the probability mechanism (t.f.p.m.) of insurance and transition functions of control mechanism (t.f.c.m.) respectively.

Under certain mild regularity conditions on the measurable spaces $(\mathrm{W}, \mathcal{W})$ and $(\mathrm{U}, \mathcal{U})$ the initial distribution $\pi_0(\cdot)$ and the families $\pi_k(\cdot; \cdot)$, $k = 1, 2, \ldots$, and $\gamma_k(\cdot; \cdot)$, $k = 0, 1, \ldots$, define over the elementary state space $(\Omega, \mathcal{F})$ a random sequence (W_k, U_k), $k = 0, 1, \ldots$, having finite-dimensional distributions

$$\mathbf{P}^{\pi, \gamma}\{W_0 \in A_0, U_0 \in B_0, \ldots, W_n \in A_n, U_n \in B_n\} = \int_{A_0} \pi_0(d\mathfrak{w}_0)$$

$$\times \int_{B_0} \gamma_0(\mathfrak{w}_0; d\mathfrak{u}_0) \ldots \int_{A_n} \pi_n(\mathcal{Y}_{n-1}, \mathfrak{u}_{n-1}; d\mathfrak{w}_n) \int_{B_n} \gamma_n(\mathcal{Y}_n; d\mathfrak{u}_n).$$

Singleperiodic (annual generic) Poisson–Exponential risk model

Poisson–Exponential (or classical) risk model: the risk reserve at time t is

$$R_s(u, c, \tau) = u + (1 + \tau)c \cdot s - V_s, \quad V_s = \sum_{i=1}^{N(s)} Y_i, \quad 0 \leqslant s \leqslant t,$$

where u is the initial risk reserve, c is the risk premium rate, τ is the adaptive premium loading, t is the year duration, $\{T_i\}_{i \geqslant 1}$ and $\{Y_i\}_{i \geqslant 1}$ are i.i.d. and mutually independent, where T_i are the interclaim times and Y_i are the amounts of claims, exponentially distributed with parameters $\lambda > 0$ and $\mu > 0$, respectively, $N(t)$ is the largest n for which $\sum_{i=1}^{n} T_i \leqslant t$ (we put $N(t) = 0$ if $T_1 > t$).

Note that

$$\mathsf{E}V_s = \frac{\lambda}{\mu} s, \quad s \geqslant 0,$$

so the premium rate $c = \dfrac{\lambda}{\mu}$ (when λ and μ are known) is calculated according to the Equity (Expected value) Principle.

Singleperiodic (annual generic) diffusion risk model

Diffusion risk model:
$$R_t(u, \tau) = u + (1 + \tau)\mu t - V(t), \quad V(t) = \mu t + \sigma \mathsf{W}_t, \quad t \geqslant 0,$$
where u is the initial risk reserve, W_t is a standard Brownian motion, μ is the premium rate calculated according to the expected value principle, i.e., $\mathsf{E}V(s) = \mu s$, τ is the adaptive premium loading, and $\sigma > 0$ is a constant diffusion coefficient, $\mathsf{D}V(s) = \sigma^2 s$. Put
$$M_t(u, \tau) = \inf_{0 < s \leqslant t} R_s(u, \tau).$$

The couple (R_t, M_t) is taken generic for the **state variable** which describes insurer's annual financial experience.

The couple (u, τ) generates two-dimensional **control variable**.

Rigorous definition of a model and synthesis of the adaptive control rules satisfying the desirable performance criteria is the central problem.

4. Synthesis of adaptive control rules in generic models

DEFINITION 1. The **target value** $u_{\mu,\sigma}(\alpha, t)$ of the risk reserve corresponding to a level $0 < \alpha < 1$ is a positive solution of the equation

$$\psi_t(u; 0) = \mathsf{P}\{M_t(u, 0) < 0\} = \alpha. \tag{1}$$

Assuming that duration of the incoming year is t, we mean by z a deviation, either positive or negative, of the past-year-end risk reserve from $u_{\mu,\sigma}(\alpha, t)$. Case $z < 0$ means deficit, case $z > 0$ means surplus.

DEFINITION 2. The strategy[1]

$$u_{z,t} = u_{\mu,\sigma}(\alpha, t) + z \quad \text{and} \quad \tau_{z,t} = -\frac{z}{\mu t}, \quad z \in \mathsf{R}, \tag{2}$$

is called the **basic adaptive** strategy.

For $0 < \alpha < 1$, introduce $c_\alpha = \Phi^{-1}(1 - \alpha/2)$.

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[1]Since the initial capital $u_{z,t}$ may not be negative, $z > -u_{\mu,\sigma}(\alpha, t) = -\sigma\sqrt{t}\, c_\alpha$. Bear it in mind when put $z \in \mathsf{R}$ for simplicity.

THEOREM 1. *For $0 < \alpha < 1$, the solution of equation* (1) *may be written as*

$$u_{\mu,\sigma}(\alpha, t) = \sigma\sqrt{t}\, c_\alpha. \tag{3}$$

THEOREM 2. *For $z \in (-\sigma\sqrt{t}\, c_\alpha, \infty)$ and for the control strategy* (2), *one has*

$$\psi_t(u_{z,t}; \tau_{z,t}) = \mathsf{P}\big\{ M_t(u_{z,t}, \tau_{z,t}) < 0 \big\}$$

$$= 1 - \Phi(c_\alpha) + \exp\left\{ 2\frac{z}{\sigma\sqrt{t}}\left(c_\alpha + \frac{z}{\sigma\sqrt{t}} \right) \right\} \Phi\left(-2\frac{z}{\sigma\sqrt{t}} - c_\alpha \right). \tag{4}$$

THEOREM 3. *For $z \in (-\sigma\sqrt{t}\, c_\alpha, \infty)$ and for the control strategy* (2), *the probability*

$$\psi_t(u_{z,t}; \tau_{z,t}) = \mathsf{P}\big\{ M_t(u_{z,t}, \tau_{z,t}) < 0 \big\},$$

regarded as a function of z, is monotone decreasing, as z increases.

DEFINITION 3. The **lower alarm barrier** of a zone with target value $u_{\mu,\sigma}(\alpha, t)$ and with level β of probability of ruin, $0 < \alpha < \beta < 1$, is $\underline{u}_{\mu,\sigma}(\alpha, \beta, t) = u_{\mu,\sigma}(\alpha, t) + z_{\mu,\sigma}(\alpha, \beta, t)$, where $z_{\mu,\sigma}(\alpha, \beta, t) < 0$ is a solution of the equation

$$\psi_t(u_{z,t}; \tau_{z,t}) = \mathsf{P}\big\{M_t(u_{z,t}, \tau_{z,t}) < 0\big\} = \beta. \tag{5}$$

THEOREM 4. *For $0 < \alpha < 1$, the solution of equation* (5) *may be written as*

$$z_{\mu,\sigma}(\alpha, \beta, t) = -\sigma\sqrt{t}\, x_{\alpha,\beta},$$

where $x_{\alpha,\beta} > 0$ is a unique root of the equation

$$1 - \Phi(c_\alpha) + \exp\{-2x(c_\alpha - x)\}\Phi(2x - c_\alpha) = \beta. \tag{6}$$

TABLE 1. Values of $x_{\alpha,\beta}$ calculated numerically using (6).

	$\beta = 110\alpha\%$	$\beta = 120\alpha\%$	$\beta = 130\alpha\%$	$\beta = 140\alpha\%$
$\alpha = 0.1$ $c_\alpha = 1.645$	$x_{\alpha,\beta} = 0.7121$	$x_{\alpha,\beta} = 0.7522$	$x_{\alpha,\beta} = 0.7897$	$x_{\alpha,\beta} = 0.8249$
$\alpha = 0.05$ $c_\alpha = 1.960$	$x_{\alpha,\beta} = 0.8360$	$x_{\alpha,\beta} = 0.8736$	$x_{\alpha,\beta} = 0.9090$	$x_{\alpha,\beta} = 0.9422$
$\alpha = 0.01$ $c_\alpha = 2.576$	$x_{\alpha,\beta} = 1.0754$	$x_{\alpha,\beta} = 1.1098$	$x_{\alpha,\beta} = 1.1423$	$x_{\alpha,\beta} = 1.1730$

DEFINITION 4. Assume that $0 < \alpha < \beta < 1$. The strategy

$$\widehat{u}_{z,t} = \begin{cases} \underline{u}_{\mu,\sigma}(\alpha, \beta, t), & z < z_{\mu,\sigma}(\alpha, \beta, t), \\ u_{\mu,\sigma}(\alpha, t) + z, & z_{\mu,\sigma}(\alpha, \beta, t) \leqslant z \leqslant 0, \\ u_{\mu,\sigma}(\alpha, t), & z > 0 \end{cases} \qquad (7)$$

and

$$\widehat{\tau}_{z,t} = \begin{cases} \overline{\tau}_{\mu,\sigma}(\alpha, \beta, t), & z < z_{\mu,\sigma}(\alpha, \beta, t), \\ -\dfrac{z}{\mu t}, & z_{\mu,\sigma}(\alpha, \beta, t) \leqslant z \leqslant 0, \\ 0, & z > 0, \end{cases} \qquad (8)$$

where

$$\overline{\tau}_{\mu,\sigma}(\alpha, \beta, t) = -\frac{z_{\mu,\sigma}(\alpha, \beta, t)}{\mu t},$$

is called **zone-adaptive** control strategy.

THEOREM 5. *For $0 < \alpha < \beta < 1$, the zone-adaptive control strategy in the diffusion model is*

$$
\widehat{u}_{z,t} = \begin{cases} \sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta}), & z < -\sigma\sqrt{t}\,x_{\alpha,\beta}, \\ \sigma\sqrt{t}\,c_\alpha + z, & -\sigma\sqrt{t}\,x_{\alpha,\beta} \leqslant z \leqslant 0, \\ \sigma\sqrt{t}\,c_\alpha, & z > 0 \end{cases} \tag{9}
$$

and

$$
\widehat{\tau}_{z,t} = \begin{cases} \dfrac{\sigma}{\mu\sqrt{t}} x_{\alpha,\beta}, & z < -\sigma\sqrt{t}\,x_{\alpha,\beta}, \\ -\dfrac{z}{\mu t}, & -\sigma\sqrt{t}\,x_{\alpha,\beta} \leqslant z \leqslant 0, \\ 0, & z > 0. \end{cases} \tag{10}
$$

DEFINITION 5. For the strategy (7)–(8), the random variable

$$S_{z,t} = \begin{cases} 0, & \underline{u}_{\mu,\sigma}(\alpha, \beta, t) \leqslant R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) \leqslant u_{\mu,\sigma}(\alpha, t), \\ R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) - u_{\mu,\sigma}(\alpha, t), & R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) > u_{\mu,\sigma}(\alpha, t), \\ -(\underline{u}_{\mu,\sigma}(\alpha, \beta, t) - R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t})), & R_t(\widehat{u}_{z,t}, \widehat{\tau}_{z,t}) < \underline{u}_{\mu,\sigma}(\alpha, \beta, t) \end{cases}$$

is called **annual excess** (of either sign) **of capital**.

5. Performance of the adaptive control strategies

▶ NO "GLOBAL OPTIMALITY CRITERION" (A FUNCTIONAL) IS FORMULATED

Targeting, or ability to keep the risk reserve inside a strip zone associated with the target value $u_{\mu,\sigma}(\alpha, t)$, is the central property of zone-adaptive strategies.

THEOREM 6. *In the* **homogeneous** *multiperiodic diffusion risk model, for the basic and the zone-adaptive strategies and for each $k = 1, 2, \ldots$,*

$$\mathbf{E}^{\pi,\gamma}\{\text{capital at the end of year } k\} = \sigma\sqrt{t}\, c_\alpha.$$

Solvency. The following result is fundamental.

THEOREM 7. *In the* **homogeneous** *multiperiodic diffusion risk model, for the zone-adaptive (α, β) strategy and for each $k = 1, 2, \ldots$,*

$$\mathbf{P}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant \beta.$$

THEOREM 8. *In the assumptions of Theorem 7, for each integer n,*

$$\mathbf{P}^{\pi,\gamma}\{\text{ruin within } n \text{ years}\} = \sum_{k=1}^{n} \mathbf{P}^{\pi,\gamma}\{\text{first ruin in year } k\} \leqslant n\beta.$$

Dynamic solvency provisions. Bearing in mind Definition 5, consider

$$\mathbf{E}^{\pi,\gamma} S_n = \sum_{k=1}^{n} \mathbf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1 \rangle}),t)},$$

where

$$\mathbf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1 \rangle}),t)} = \int_{\mathrm{W}} P(\mathfrak{w}_0, d\mathfrak{w}_1) \dots \int_{\mathrm{W}} P(\mathfrak{w}_{k-2}, d\mathfrak{w}_{k-1})$$

$$\times \left(\int_{\{\mathfrak{w}_k^{\langle 1 \rangle} > \sigma\sqrt{t}\, c_\alpha\}} (\mathfrak{w}_k^{\langle 1 \rangle} - \sigma\sqrt{t}\, c_\alpha) P(\mathfrak{w}_{k-1}, d\mathfrak{w}_k^{\langle 1 \rangle} \times \{0,1\}) \right.$$

$$\left. - \int_{\{\mathfrak{w}_k^{\langle 1 \rangle} < \sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta})\}} (\sigma\sqrt{t}\,(c_\alpha - x_{\alpha,\beta}) - \mathfrak{w}_k^{\langle 1 \rangle}) P(\mathfrak{w}_{k-1}, d\mathfrak{w}_k^{\langle 1 \rangle} \times \{0,1\}) \right).$$

THEOREM 9. *In the* **homogeneous** *multiperiodic diffusion risk model, for the zone-adaptive (α, β) strategy and for each $k = 1, 2, \dots,$*

$$\mathbf{E}^{\pi,\gamma} S_{(z(W_{k-1}^{\langle 1 \rangle}),t)} > 0.$$

A conceptual emphasis

▶ NO "GLOBAL OPTIMALITY CRITERION" (A FUNCTIONAL) IS FORMULATED

Single-purposed objectives like "to find the policy which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy" may appear deficient to practical people. That kind of objectives may impress some shareholders, let alone mathematicians, but it will be resented by other parties to the insurance business.

C.-O. Segerdhal in his discussion of the paper [Borch, K. (1967)] expressed it by means of a grotesque paradox:

> "I should think that if a manager of an insurance company came to his board or to his policyholders and said something like this: "Gentlemen, I am running this company along lines proposed by modern economics. This means that the company will certainly go broke. The probability of ruin is equal to one. It will go broke, but I will try to postpone as long as possible the deplorable but inevitable moment when you lose your money. Or, alternatively, before that happens, I will try to make as much money as possible to distribute. I do not care what happens then", I think such a managing director would not need any *deus ex machina* to be relieved of the burden of his duties. His board would see to that immediately."