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SCENARIO ANALYSIS FOR A MULTIPERIOD DIFFUSION MODEL OF RISK

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1. Introduction: underwriting process cyclic due to random surrounding

Long-term variations called "business cycles", are typically common for the most insurers and have several potential causes.

Understanding the driving forces of the underwriting cycles is a paramount theoretical and important practical problem.

▶ Emphasize is put on cycles (cyclic behavour) attributed to the fluctuations due to random surroundings, to volatile interest rates, or to random up- and down-swings of the risk exposure in the portfolio. Deficiencies are introduced by the exterior ambiguities limited by the so-called *scenarios of nature*.

• Such fluctuations can not be foreseen and their dynamics is known deficiently since its origin used to be exogenous with respect to the insurance industry.

• It causes inevitable errors in the rate making, and irregularly cyclic underwriting process ensues.

• Adaptive control strategies fighting back cycles due to scenarios of nature are proposed in the multiperiod framework.

2. A simplistic model of insurance process and a volatile scenario of nature

$$\mathfrak{w}_0 \underbrace{\xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \mathfrak{w}_{k-1} \underbrace{\xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \cdots .$$

- the state variables $\mathfrak{w}_k$,
- the control variables $\mathfrak{u}_k$,
- the control rules γ_{k-1},
- the probability mechanisms of insurance π_k.

• Assume that the annual probability mechanism of insurance π_k is induced by the claim out-pay process $V_s(M) = Ms + \sigma(M)\mathsf{W}_s$, $0 \leqslant s \leqslant t$, which yields the annual risk reserve process as

$$R_s(u, c, M) = u + cs - V_s(M), \quad 0 \leqslant s \leqslant t,$$

where u is the initial risk reserve, c is the premium intensity, M is the random claim out-pay rate, $\sigma(\cdot)$ is a known function assuming positive values and $\sigma^2(M)$ is the random volatility; W_s, $0 \leqslant s \leqslant t$, is the standard Brownian motion.

• Development in time: introduce the sequence $\{\mathsf{W}_s^{[k]}, 0 \leqslant s \leqslant t\}$, $k = 1, 2, \ldots$, of *independent* Brownian motions and the sequence M_k, $k = 1, 2, \ldots$, of the random claim intensities. Assume that these sequences are *independent* of each other.

• The annual claim out-pay processes are $V_s^{[k]}(M_k)$, $k = 1, 2, \ldots$.

• By volatile (homogeneous and with known generic risk) scenario of nature associated with the multi-period model and the annual mechanisms of insurance we mean the sequence of i.i.d. claim intensities M_k, $k = 1, 2, \ldots$, with known generic distribution G with support $\mathrm{M} = [\mu_{\min}, \infty)$, $0 < \mu_{\min} < \infty$, i.e., only the lower bound $\mu_{\min}$ of the claim intensity, or the *most favorable* case for the insurer, is a priori known.

• The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, satisfies the α-level integral solvency criterion if

$$\sup_{w>0} \mathsf{P}\Big\{ \inf_{0 \leqslant s \leqslant t} R_s(u(w), c(w), M) < 0 \Big\} = \sup_{w>0} \int_{\mathrm{M}} \boldsymbol{\psi}_t(u(w), c(w), m)\, G(dm) \leqslant \alpha.$$

• The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, satisfies the (α_1, α_2)-solvency criterion with $\alpha_i \in (0, 1/2)$, $i = 1, 2$, if for the $(1 - \alpha_1)$-quantile μ_{α_1} of c.d.f. G

$$\sup_{w>0, m \leqslant \mu_{\alpha_1}} \boldsymbol{\psi}_t(u(w), c(w), m) \leqslant \alpha_2.$$

• The adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion sharply if

$$\boldsymbol{\psi}_t(u(w), c(w), \mu_{\alpha_1}) = \alpha_2$$

for all $w > 0$.

THEOREM 1. *Assume that the adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion. Then it satisfies the $(\alpha_1 + \alpha_2)$-level integral solvency criterion.*

3. Synthesis of the annual adaptive controls

• For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, and for the $(1 - \alpha_1)$-quantile μ_{α_1} of c.d.f. G the solution $u_{\alpha_2,t}(c, \mu_{\alpha_1})$ of the equation

$$\boldsymbol{\psi}_t(u, c, \mu_{\alpha_1}) = \mathsf{P}\Big\{ \inf_{0 \leqslant s \leqslant t} R_s(u, c, M) < 0 \mid M = \mu_{\alpha_1} \Big\} = \alpha_2$$

with respect to u is called α_2-level initial capital corresponding to the claim intensity μ_{α_1} and to the premium intensity c.

• The solution $c_{\alpha_2,t}(u, \mu_{\alpha_1})$ with respect to c is called α_2-level premium intensity corresponding to the claim intensity μ_{α_1} and to the initial capital u.

• By definition, $c_{\alpha_2,t}(u_{\alpha_2,t}(c, \mu_{\alpha_1}), \mu_{\alpha_1}) = c$, $u_{\alpha_2,t}(c_{\alpha_2,t}(u, \mu_{\alpha_1}), \mu_{\alpha_1}) = u$.

▶ The "fair" long-time average premium rate is $\mathsf{E}M$ since

$$\mathsf{E}V_t(M) = \mathsf{E}M \cdot t,$$

so that the average annual claim amount is equal to the total annual premiums.

• We name ***equitable*** those controls $(u(w), c(w))$ which are holding the risk reserve large enough to secure solvency, but at the expectation i.e., around the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$. Otherwise, one would rightfully argue that *this provision is used to cover the unexpected*.

• The adaptive control $(u(w), c(w))$, where w is the past-year-end capital, is called ultimately equitable[1], if

$$\mathsf{E}R_t(u(w), c(w), \mu_{\alpha_1}) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

uniformly in $w \in \mathsf{R}^+$.

• Equity requires premiums well-balanced with claims. Insureds ought to pay premiums which are sensibly concentrated around the long-run mean value of their losses. In that sense the customers will not be overcharged, but only in the long run (i.e., in the average throughout several insurance years), while in the separate insurance years the premiums may be above or below average.

[1] It may be also called balanced around the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$, or targeted at that "fair" capital value.

For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, the adaptive control <u>sensitive to</u> w, is

$$\hat{u}(w) = \begin{cases} u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ w, & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases}$$

$$\hat{c}(w) = \begin{cases} c_{\min}, & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ c_{\alpha_2,t}(w, \mu_{\alpha_1}), & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ c_{\max}, & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases}$$

where $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$.

THEOREM 2. *The control* $(\hat{u}(w), \hat{c}(w))$ *satisfies the* (α_1, α_2)*-solvency criterion sharply and, consequently, satisfies the* $(\alpha_1 + \alpha_2)$*-level integral solvency criterion.*

A technical drawback of the control $(\hat{u}(w), \hat{c}(w))$ is the necessity to calculate $c_{\alpha_2,t}(w, \mu_{\alpha_1})$ for each w, i.e., to determine that non-linear function as a whole. Introduce

$$\bar{\tau}_{\alpha_2,t}(w) = -\frac{w - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t},$$

where $\mathsf{E}M$ is the ultimately equitable, or "fair" premium rate. Consider the control with linearized adaptive premium rates,

$$\bar{u}(w) = \begin{cases} u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ w, & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases}$$

$$\bar{c}(w) = \begin{cases} \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})), & w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})), & 0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), \end{cases}$$

where $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$.

• Construct a control with linear adaptive loading, but free of the drawback of uncontrollable solvency (i.e. improve $(\bar{u}(w), \bar{c}(w))$).

• For the level β such that

$$0 < \alpha_2 \leqslant \beta < 1/2,$$

introduce the strip zone with the lower bound $\underline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + z_{\beta,t}$, where $z_{\beta,t} < 0$ is a solution of the equation

$$\boldsymbol{\psi}_t\big(z + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), \mathsf{E}M - \frac{z}{t}, \mu_{\alpha_1}\big) = \beta \qquad (1)$$

with respect to z, and with a certain upper bound $\overline{u}_{\beta,t}$ such that

$$u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant \underline{u}_{\beta,t} \leqslant u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) \leqslant \overline{u}_{\beta,t} \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}).$$

► Eq. (1) has a unique solution $z_{\beta,t} < 0$ and the explicit expression for $z_{\beta,t}$ is obtained.

• There are different ways to select the upper bound $\overline{u}_{\beta,t}$. For example (recall that $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$), one may take $\overline{u}_{\beta,t} = u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, or[2] $\overline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$.

[2] That selection is sensible because the premiums will not be larger than $\mathsf{E}M$ (i.e., $\overline{\mu}_{\beta,t} = \mathsf{E}M$ in (2)), and no capital exceeding one least necessary to guarantee the non-ruin with probability α_2 is "frozen" as solvency reserve. For $\overline{u}_{\beta,t}$ selected in that way, $|z_{\beta,t}|$ is the width of the strip zone. These reasons may be however unconvincing for a decision maker with other preferences.

<u>Zone-adaptive</u> annual control with <u>linearized premiums</u> is

$$
\begin{aligned}
\widehat{u}(w) &= \begin{cases} \overline{u}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ w, & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{u}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t}, \end{cases} \\
\widehat{c}(w) &= \begin{cases} \overline{\mu}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{\mu}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t}, \end{cases}
\end{aligned}
\tag{2}
$$

where

$$
\overline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\overline{u}_{\beta,t} - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t},
$$

$$
\underline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\underline{u}_{\beta,t} - u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})}{t} = \mathsf{E}M - \frac{z_{\beta,t}}{t}.
$$

THEOREM 3. *For* $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, *the control* $(\widehat{u}(w), \widehat{c}(w))$ *is ultimately equitable and satisfies the* (α_1, β)*-solvency criterion sharply.*

4. Multi-period model of risk under volatile scenario

• General multiperiodic insurance process with annual accounting and annual control interventions

$$\underbrace{\mathfrak{w}_0 \xrightarrow{\gamma_0} \mathfrak{u}_0 \xrightarrow{\pi_1} \mathfrak{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \underbrace{\mathfrak{w}_{k-1} \xrightarrow{\gamma_{k-1}} \mathfrak{u}_{k-1} \xrightarrow{\pi_k} \mathfrak{w}_k}_{k\text{-th year}} \cdots .$$

• Write $\mathbf{P}^{\pi,\gamma}\{\cdot\}$ for the Markov chain with transition probability P. For brevity, we denote by $\mathbf{P}^{\pi,\gamma}_{\boldsymbol{m}}\{\cdot\}$ the conditional distribution $\mathbf{P}^{\pi,\gamma}\{\cdot \mid \boldsymbol{M} = \boldsymbol{m}\}$, where $\boldsymbol{M} = \{M_k,\ k = 1, 2, \dots\} \in \mathbf{M} = \mathrm{M}^\infty$ is the sequence of i.i.d. random variables and $\boldsymbol{m}$ is its realization. Evidently, $\mathbf{P}^{\pi,\gamma}_{\boldsymbol{m}}\{\cdot\}$ corresponds to the case when the trajectory $\boldsymbol{m}$ of the scenario of nature is fixed.

$$\mathbf{P}^{\pi,\gamma}\left\{\begin{array}{c}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{array}\right\} = \int_{\mathsf{R}\times\mathrm{M}} G(dm_1) P_{m_1}(w; d\mathfrak{w}_1^{\langle 1\rangle} \times \{0\}) \dots$$

$$\dots \int_{\mathsf{R}\times\mathrm{M}} G(dm_{k-1})\, P_{\mu_{k-1}}(\mathfrak{w}_{k-2}^{\langle 1\rangle}; d\mathfrak{w}_{k-1}^{\langle 1\rangle} \times \{0\}) \int_{\mathsf{R}\times\mathrm{M}} G(dm_k)\, P_{m_k}(\mathfrak{w}_{k-1}^{\langle 1\rangle}; \mathsf{R} \times \{1\}),$$

5. Conclusions

THEOREM 4 (Solvency). *In the homogeneous multi-period diffusion model with starting capital $w \in \mathsf{R}^+$, for the homogeneous pure Markov strategy generated by the annual control $(\hat{u}(w), \hat{c}(w))$,*

$$\sup_{w \in \mathsf{R}^+} \mathbf{P}^{\pi;\gamma} \left\{ \begin{array}{c} \text{first ruin in year } k, \\ \text{as starting capital is } w \end{array} \right\} \leqslant \alpha_1 + \alpha_2, \quad k = 1, 2, \ldots .$$

For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums $(\overset{\frown}{u}(w), \overset{\frown}{u}(w))$,

$$\sup_{w \in \mathsf{R}^+} \mathbf{P}^{\pi;\gamma} \left\{ \begin{array}{c} \text{first ruin in year } k, \\ \text{as starting capital is } w \end{array} \right\} \leqslant \alpha_1 + \beta, \quad k = 1, 2, \ldots .$$

THEOREM 5 (Equity). *For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums $(\overset{\frown}{u}(w), \overset{\frown}{u}(w))$,*

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\pi;\gamma} \left(\begin{array}{c} \textit{capital at the end of year } k, \\ \textit{as starting capital is } w \end{array} \right) \right] = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}).$$